Contribution to the Chebyshev Approximations of the Normalized Low-Pass Prototype

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Abstract. *The standard approximation algorithms are well described in the literature, but some equiripple approximations are described with some deficiencies. Especially Chebyshev and inverse Chebyshev approximations are often wrongly interpreted or implemented. In this paper, we propose all formulas for computing Chebyshev approximations in a standard form. Transformations, which are necessary for circuit implementation, are presented in the analytical form too.*

Keywords

Analog filters, approximation, filter design.

1. Introduction

Approximations of the normalized low-pass prototypes in the Chebyshev sense play an important role among equiripple approximations. These approximations offer an equiripple curve of the module frequency characteristics in the pass-band or in the stop-band (or in both). In the literature, these approximation methods are called Chebyshev approximation and inverse Chebyshev approximation.

Chebyshev approximations are discussed in this paper with the following intention:

- In the case of even-order transfer functions (Chebyshev approximation), the LC lattice filters cannot be realized with equal termination resistors $(r_1 \neq r_2 \neq 1)$. Tables with selected values of ripple of the NLP (*normalized low-pass prototype*) in the pass-band are printed in the Saal catalogue. Transformation formulas, which can be used to the termination with equal resistors $r_1 = r_2 = 1$, are printed in this catalogue also. Nevertheless, general analytic formulas for an arbitrary requirement to the error in the pass-band are missing in the literature.
- In the literature, inverse Chebyshev approximation is not often described or this approximation is wrongly interpreted: the older implementation in MATLAB can serve as an example: the transfer function of the NLP is here normalized at the stop-band edge $\Omega = 1$; however this frequency is assigned with the pass-band edge of the NLP in the standard case.
- Even order transfer functions (inverse Chebyshev transformation) cannot be realized with classic LC lattice structures. These filters can be realized only with bound inductors (inductors with junctions).

Presentation of the correct formulas for transformation of the NLP transfer functions is a main acquisition of this paper. These transformed transfer functions can be realized with classic lattice LC structures with equal termination resistors $r_1 = r_2 = 1$ or with lattice LC structures with an open termination or a short one. Algorithms of circuit realizations are described in [1], [3], e. g.

2. Order of the Approximation

Approximations are based on the characteristic equation:

$$H(s)\, H(-s)\, \big|\, s = j\Omega = \left|H(j\Omega)\right|^2 = \\ = \frac{1}{1+\varepsilon^2 \varphi^2(\Omega)}, \tag{1}$$

where $H(s)$ is a transfer function and $\varphi(\Omega)$ is a characteristic function. Requirements on selectivity of the NLP are given from the filter specification schema. Filter specifications are shown in Fig. 1a. The parameter a_p describes an input tolerance in decibels [dB] on pass-band, i.e. in the frequency interval $\Omega \in <0, 1>$.

The stop-band is defined by the boundary frequency Ω_s and by the acceptable error of the approximation in the stop-band a_s. Parameters a_p, a_s, 1, Ω_s are the input parameters of the approximation problem of a NLP synthesis. These requirements, in the defiance of the chosen approximation type, can be described by secondary parameters ε, k, k_1. The constant ε describes the maximum acceptable error (ripple) of the module frequency characteristic in the pass-band, the parameter k describes the attitude between boundary frequencies of the pass-band and stop-band. The parameter k_1 describes the distance of the transfer module in the pass-band and stop-band. A relation between secondary parameters and the filter specifications is shown in

Fig. 1b. Calculation of the order of NLP with Chebyshev and inverse Chebyshev module frequency characteristics is listed in Tab. 1.

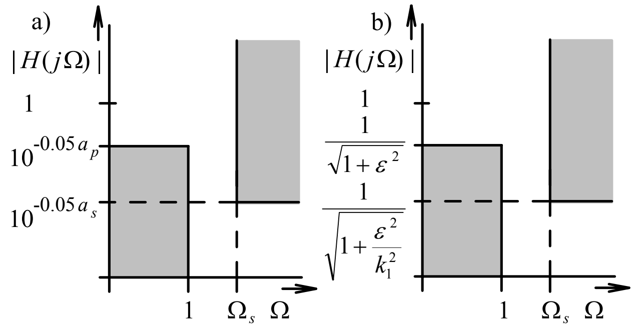

Fig. 1. Transfer specifications of NLP.

The order of the approximation has to be an integer number. Formulas for the calculation of secondary parameters are given in Tab. 1. Parameters k_1 and a_s are newly recalculated in the first case (k and Ω_s are unchanged), parameters k and Ω_s are newly recalculated in the second case (k_1 and a_s are unchanged).

Input parameters	a_p, a_s, 1, Ω_s
Secondary parameters of NLP	$\varepsilon = \sqrt{10^{0.1a_p} - 1}$ $k = 1/\Omega_s$ $k_1 = \sqrt{\dfrac{10^{0.1a_p} - 1}{10^{0.1a_s} - 1}}$
The order of a filter	$n \geq \dfrac{\arg\cosh(1/k)}{\arg\cosh(1/k_1)}$
Converted parameters (*n* is an integer)	
The 1st option	$k_1 = \dfrac{1}{\cosh[n \arg\cosh(1/k)]}$ $a_s = 10 \log\left(1 + \dfrac{\varepsilon^2}{k_1^2}\right)$
The 2nd option	$k = \dfrac{1}{\cosh[(1/n) \arg\cosh(1/k_1)]}$ $\Omega_s = 1/k$

Tab. 1. Evaluation of the order of approximation.

3. Chebyshev Approximation

The order of the approximation n and new secondary parameters are input parameters of the approximation. Calculation of new secondary parameters is listed in Tab. 1. The method of the retrieval of the transfer function $H(s)$ and the retrieval of the characteristic function parameters is listed in Tab. 2. If n is an odd number, we can realize the filter prototype with an equal termination resistors $r_1 = r_2 = 1$ from functions $H(s)$ and $\varphi(s)$. The filter prototype can be realized with the classical LC lattice structure with unequal termination resistors $r_1 \neq r_2 \neq 1$ in the case of the even order of the transfer function. If we can realize a filter prototype with equal termination resistors, we have to use the transformation of the even order transfer function:

$$s_B^2 = \frac{s^2 + \tilde{\Omega}_{01}^2}{1 - \tilde{\Omega}_{01}^2} \ . \tag{2}$$

The transformation is applied to the original zeros of the characteristic function $\tilde{\Omega}_{0\mu}$ and poles $s_\mu = \alpha_\mu \pm j\beta_\mu$ of the transfer function. New functions $H_B(s)$ and $\varphi_B(s)$ are results of the transformation and these functions are listed in Tab. 3. Maximum of the transfer module at the lowest frequency $\tilde{\Omega}_{0\mu}$ is moved to the beginning according to Fig. 2 when using the transformation (2). The curve like curve in the odd case is obtained with this approximation procedure. The degradation of the selectivity is shown in Fig. 2.

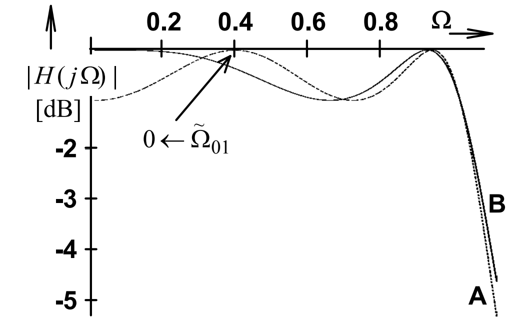

Fig. 2. Transformation of the transfer maximum $\tilde{\Omega}_{01}$ at the lowest frequency to zero.

4. Inverse Chebyshev Approximation

In this case, input parameters are the order of the approximation and new secondary parameters. These parameters are recalculated on n–integer according to Tab. 1. The approximation process, i.e. the retrieval of the function $H(s)$ and $\varphi(s)$, is shown in Tab. 4. In the case of the even order of n, the transfer function is of a finite value on $\Omega \to \infty$, and this transfer function can be realized with the bound inductors only.

We have to use the transformation of the transfer zeros (poles of the characteristic function) $\Omega_{0\mu}$ and transfer poles $s_\mu = \alpha \pm j\beta$ if we can realize a filter prototype without the bound inductors. The transformation formula

$$s_C^2 = \frac{\left(\Omega_{01}^2 - 1\right)s^2}{\Omega_{01} + s^2} \tag{3}$$

is applied on zeros of the initial transfer function in Tab. 5.

Impact of the transformation on the module frequency characteristics is shown in Fig. 3. This case can be realized with a classic LC lattice structure. Using transform (3), selectivity of filter is lower and stop-band edge is shifted.

Zeros of the characteristic function	$\tilde{\Omega}_{0\mu} = \cos\frac{(2\mu-1)\pi}{2n}, \quad \mu = 1,2,\ldots,n$
Additional parameters	$a = \frac{1}{2}\left(\left(\sqrt{1+\frac{1}{\varepsilon^2}}+\frac{1}{\varepsilon}\right)^{1/n} - \left(\sqrt{1+\frac{1}{\varepsilon^2}}-\frac{1}{\varepsilon}\right)^{1/n}\right)$ $b = \frac{1}{2}\left(\left(\sqrt{1+\frac{1}{\varepsilon^2}}+\frac{1}{\varepsilon}\right)^{1/n} + \left(\sqrt{1+\frac{1}{\varepsilon^2}}-\frac{1}{\varepsilon}\right)^{1/n}\right)$
Poles of the transfer function	$s_\mu = \alpha_\mu + j\beta_\mu = -a\sin\frac{(2\mu-1)\pi}{2n} + jb\cos\frac{(2\mu-1)\pi}{2n}$ $\mu = 1,2,\ldots,n$
The transfer function *(n* is an odd number)	$H(s) = \frac{H_0}{(s+a)\prod_{\mu=1}^{m}\left(s^2 - 2\alpha_\mu s + \alpha_\mu^2 + \beta_\mu^2\right)}$ $H_0 = a\prod_{\mu=1}^{m}\left(\alpha_\mu^2 + \beta_\mu^2\right) = \frac{1}{\varepsilon 2^m}, \quad m = \frac{n-1}{2}$
The transfer function *(n* is an even number)	$H(s) = \frac{H_0}{\prod_{\mu=1}^{m}\left(s^2 - 2\alpha_\mu s + \alpha_\mu^2 + \beta_\mu^2\right)}$ $H_0 = \frac{1}{\sqrt{1-\varepsilon^2}}\prod_{\mu=1}^{m}\left(\alpha_\mu^2 + \beta_\mu^2\right) = \frac{1}{\varepsilon 2^{m-1}}, \quad m = \frac{n}{2}$
The characteristic function *(n* is an odd number)	$\varphi(s) = 2^m s\prod_{\mu=1}^{m}\left(s^2 + \tilde{\Omega}_{0\mu}^2\right) \quad , \ m = \frac{n-1}{2}$
The characteristic function *(n* is an even number)	$\varphi(s) = 2^{m-1} s\prod_{\mu=1}^{m}\left(s^2 + \tilde{\Omega}_{0\mu}^2\right) \quad , \ m = \frac{n}{2}$

Tab. 2. Chebyshev approximation of the NLP.

Input parameters (Tab. 2.)	$\tilde{\Omega}_{0\mu} \quad , \ s_\mu = \alpha_\mu + j\beta_\mu$
Additional parameters	$A_\mu = \alpha_\mu^2 - \beta_\mu^2 + \tilde{\Omega}_{01}^2 \quad , \ B_\mu = 2\alpha_\mu\beta_\mu$
Poles of the transfer function	$s_{\mu B} = \alpha_{\mu B} + j\beta_{\mu B} =$ $= -\sqrt{\frac{A+\sqrt{A^2+B^2}}{2\left(1-\tilde{\Omega}_{01}^2\right)}} + j\sqrt{\frac{-A+\sqrt{A^2+B^2}}{2\left(1-\tilde{\Omega}_{01}^2\right)}}$
The transfer function	$H_B(s) = \prod_{\mu=1}^{m}\frac{H_{0B}}{s^2 - 2\alpha_{\mu B}s + \alpha_\mu^2 + \beta_\mu^2}$ $H_{0B} = \prod_{\mu=1}^{m}\left(\alpha_{\mu B}^2 + \beta_{\mu B}^2\right), \quad m = \frac{n}{2}$
Zeros of the transfer function	$\tilde{\Omega}_{0\mu B} = \sqrt{\frac{\tilde{\Omega}_{0\mu}^2 - \tilde{\Omega}_{01}^2}{1-\tilde{\Omega}_{01}^2}} \quad , \ \mu = 2,\ldots,n/2$
The characteristic function	$\varphi_B(s) = \varphi_{0B}s^2\prod_{\mu=1}^{m}\left(s^2 + \tilde{\Omega}_{0\mu B}^2\right) \quad , \ \varphi_{0B} = \frac{1}{\varepsilon H_{0B}}$

Tab. 3. Chebyshev approximation of the even order transfer function — transformation B.

Additional parameters	
$a = \frac{1}{2}\left(\left(\sqrt{1+\varepsilon^2 T_n^2\left(\frac{1}{k}\right)} + \varepsilon T_n\left(\frac{1}{k}\right)\right)^{1/n} + \left(\sqrt{1+\varepsilon^2 T_n^2\left(\frac{1}{k}\right)} - \varepsilon T_n\left(\frac{1}{k}\right)\right)^{1/n}\right)$ $b = \frac{1}{2}\left(\left(\sqrt{1+\varepsilon^2 T_n^2\left(\frac{1}{k}\right)} + \varepsilon T_n\left(\frac{1}{k}\right)\right)^{1/n} - \left(\sqrt{1+\varepsilon^2 T_n^2\left(\frac{1}{k}\right)} - \varepsilon T_n\left(\frac{1}{k}\right)\right)^{1/n}\right)$	
Zeros of the characteristic function	$\Omega_{0\mu} = \frac{1}{k \cdot \cos\left((2\mu-1)\frac{\pi}{2n}\right)}$
Poles of the transfer function	$s_\mu = \frac{1}{k}\frac{-a\sin(2\mu-1)\frac{\pi}{2n} + jb\cos(2\mu-1)\frac{\pi}{2n}}{a^2\sin^2(2\mu-1)\frac{\pi}{2n} + b^2\cos^2(2\mu-1)\frac{\pi}{2n}} = \alpha_\mu + j\beta_\mu, \quad \mu = 1,2,\ldots,n$
The transfer function, *n* is an even number	$H(s) = H_0 \prod_{\mu=1}^{m} \frac{s^2+\Omega_{0\mu}^2}{s^2 - 2\alpha_\mu s + \alpha_\mu^2 + \beta_\mu^2}$ $H_0 = \frac{1}{\sqrt{1+\frac{\varepsilon^2}{k_1^2}}}, \quad m = \frac{n}{2}$
The transfer function, *n* is an odd number	$H(s) = \frac{H_0}{s + \frac{1}{a \cdot k}} \prod_{\mu=1}^{m} \frac{s^2+\Omega_{0\mu}^2}{s^2 - 2\alpha_\mu s + \alpha_\mu^2 + \beta_\mu^2}$ $H_0 = \frac{nk_1}{\varepsilon k}, \quad m = \frac{n-1}{2}$
The characteristic function, *n* is an even number	$\varphi(s) = \frac{(-1)^m}{k_1} s^n \prod_{\mu=1}^{m} \frac{1}{s^2+\Omega_{0\mu}^2}, \quad m = \frac{n}{2}$
The characteristic function, *n* is an odd number	$\varphi(s) = \frac{(-1)^m k}{nk_1} s^n \prod_{\mu=1}^{m} \frac{1}{s^2+\Omega_{0\mu}^2}, \quad m = \frac{n-1}{2}$

Tab. 4. Inverse Chebyshev approximation.

5. An Example: Inverse Chebyshev Approximation

Frequency and transfer specifications of the low-pass filter are a_p = 1 dB, a_s = 38 dB, Ω_s = 2.35. Secondary filter parameters are ε = 0.508847, k = 0.425532, k_1 = 0.006406. The order of the transfer function is n = 4. We have to achieve a correction of the stop-band edge on n (n is an integer). We can recalculate new values k = 0.450318, Ω_s = = 2.220652 using formulas shown in Tab. 1. Auxiliary parameters a and b are calculated from formulas shown in Tab. 4.: a = 0.163426, b = 1.915939. Poles of the transfer function $H(s)$ are

$$s_{1,2} = \alpha_1 \pm j\beta_1 = -0.394058 \pm j\,1.115310\,,$$

$$s_{3,4} = \alpha_2 \pm j\beta_2 = -1.190119 \pm j\,0.577928\,.$$

Zeros of the transfer function are Ω_{01} = 2.403617, Ω_{02} = = 5.802844. Transfer function $H(s)$ and characteristic function $\varphi(s)$ have a form

$$H(s) = \frac{1.25893\cdot 10^{-2}\left(s^2+5.777274\right)\cdot\left(s^2+33.67300\right)}{s^4+3.168354s^3+5.025486s^2+4.709928s+2.449133}\,,$$

$$\varphi(s) = \frac{1.560911\cdot 10^2 s^4}{\left(s^2+5.777374\right)\left(s^2+33.67300\right)}\,.$$

Transformation (3) moves a zero of the function $H(s)$ at the highest frequency to infinity.

$$H_C(s) = \frac{0.340783\cdot\left(s^2+6.766801\right)}{s^4+3.147718s^3+4.954065s^2+4.612651s+2.306010}\,,$$

$$\varphi_C(s) = \frac{5.766799s^4}{s^2+6.766801}\,.$$

Input parameters (Tab. 4.)	$\Omega_{0\mu}$, $s_\mu = \alpha_\mu + j\beta_\mu$, $m = n/2$
Zeros of the transfer function $H_C(s)$	$\Omega_{0\mu C} = \Omega_{0\mu}\sqrt{\dfrac{\Omega_{01}^2 - 1}{\Omega_{0\mu}^2 - \Omega_{01}^2}}$
Additional parameters	$A_\mu = \alpha_\mu^2 - \beta_\mu^2 + \Omega_{01}^2$, $B_\mu = 2\alpha_\mu\beta_\mu$ $C_\mu = \sqrt{\dfrac{A_\mu + \sqrt{A_\mu^2 + B_\mu^2}}{2}}$ $D_\mu = \sqrt{\dfrac{-A_\mu + \sqrt{A_\mu^2 + B_\mu^2}}{2}}$
Poles of the transfer function $H_C(s)$	$s_{\mu C} = \alpha_{\mu C} + j\beta_{\mu C}$ $\alpha_{\mu C} = \sqrt{\dfrac{\Omega_{01}^2 - 1}{A_\mu^2 + B_\mu^2}}\left(\alpha_\mu C_\mu - \beta_\mu D_\mu\right)$ $\beta_{\mu C} = \sqrt{\dfrac{\Omega_{01}^2 - 1}{A_\mu^2 + B_\mu^2}}\left(\alpha_\mu D_\mu + \beta_\mu C_\mu\right)$
The transfer function $H_C(s)$ $H_C(s) = \dfrac{H_{0C}}{s^2 - 2\alpha_{mC}s + \alpha_{mC}^2 + \beta_{mC}^2}\prod_{\mu=1}^{m-1}\dfrac{s^2 + \Omega_{0\mu C}^2}{s^2 - 2\alpha_{\mu C}s + \alpha_{\mu C}^2 + \beta_{\mu C}^2}$ $H_{0C} = \left(\alpha_{mC}^2 + \beta_{mC}^2\right)\prod_{\mu=1}^{m}\dfrac{\alpha_{\mu C}^2 + \beta_{\mu C}^2}{\Omega_{0\mu C}^2}$	
The characteristic function $\varphi_C(s)$	$\varphi_C(s) = \varphi_{0C}s^n\prod_{\mu=1}^{m-1}\dfrac{1}{s^2 + \Omega_{0\mu C}^2}$, $\varphi_{0C} = \dfrac{1}{H_{0C}}$

Tab. 5. Inverse Chebyshev approximation — transformation B.

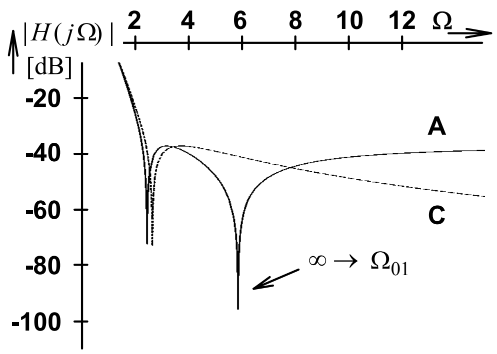

Fig. 3. Transformation of the transfer zero on the highest frequency to infinity.

6. Conclusion

Transformations of the transfer functions, which are listed in Tab.3. and Tab.5., are additions of the realization options of filters with the equiripple curve.

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